

# A THEORY OF LORENTZIAN KAC–MOODY ALGEBRAS

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*Dedicated to 90-th Anniversary of Lev S. Pontryagin*

**ABSTRACT.** We present a variant of the Theory of Lorentzian (i. e. with a hyperbolic generalized Cartan matrix) Kac–Moody algebras recently developed by V. A. Gritsenko and the author. It is closely related with and strongly uses results of R. Borcherds. This theory should generalize well-known Theories of finite Kac–Moody algebras (i. e. classical semi-simple Lie algebras corresponding to positive generalized Cartan matrices) and affine Kac–Moody algebras (corresponding to semi-positive generalized Cartan matrices).

Main features of the Theory of Lorentzian Kac–Moody algebras are: One should consider *generalized* Kac–Moody algebras introduced by Borcherds. Denominator function should be an automorphic form on IV type Hermitian symmetric domain (first example of this type related with Leech lattice was found by Borcherds). The Kac–Moody algebra is graded by elements of an integral hyperbolic lattice  $S$ . Weyl group acts in the hyperbolic space related with  $S$  and has a fundamental polyhedron  $\mathcal{M}$  of finite (or almost finite) volume and a lattice Weyl vector.

There are results and conjectures which permit (in principle) to get a “finite” list of all possible Lorentzian Kac–Moody algebras. Thus, this theory looks very similar to Theories of finite and affine Kac–Moody algebras but is much more complicated. There were obtained some classification results on Lorentzian Kac–Moody algebras and many of them were constructed.

## 0. INTRODUCTION

Lev Semenovich Pontryagin was the Great Mathematician. He had many points of interest. One of them was topological groups, Lie groups and Lie algebras. I mention his classical book “Topological Groups” [P] containing his results.

On this conference, I am glad to present the talk related with this subject. It is devoted to *Lorentzian* (or *hyperbolic*) Kac–Moody Lie algebras which are a hyperbolic analogy of classical semi-simple Lie algebras. A theory of these algebras was recently developed by V.A. Gritsenko and the author [GN1]—[GN7]. It is closely related with and strongly uses results of R. Borcherds [B1] — [B5].

## 1. LORENTZIAN KAC–MOODY ALGEBRAS

**1.1. Some general results on Kac–Moody algebras.** One can find all definitions and details in the classical book by Victor Kac [K].

A *generalized Cartan matrix*  $A$  is an integral square matrix of a finite rank which has only 2 on the diagonal and non-positive integers out of the diagonal. We shall consider only *symmetrizable* generalized Cartan matrices  $A$ . It means that there exists a diagonal matrix  $D$  with positive rational coefficients such that  $B = DA$  is

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integral and symmetric. Then  $B$  is called the *symmetrization* of  $A$ . By definition,  $\text{sign}(A) = \text{sign}(B)$ . We shall suppose that  $A$  is *indecomposable* which means that there does not exist a decomposition  $I = I_1 \cup I_2$  of the set  $I$  of indices of  $A$  such that  $a_{ij} = 0$  if  $i \in I_1$  and  $j \in I_2$ .

Each generalized Cartan matrix  $A$  defines a Kac–Moody Lie algebra  $\mathfrak{g}(A)$  over  $\mathbb{C}$ . The Kac–Moody algebra  $\mathfrak{g}(A)$  is defined by the set of generators and defining relations prescribed by the generalized Cartan matrix  $A$ . They are due to V. Kac and R. Moody. In fact, they are a natural generalization of classical results of Killing, Cartan, Weyl, Chevalley and Serre about finite-dimensional semisimple Lie algebras. One should introduce the set of generators  $h_i, e_i, f_i, i \in I$ , with defining relations

$$\begin{cases} [h_i, h_j] = 0, & [e_i, f_i] = h_i, & [e_i, f_j] = 0, \text{ if } i \neq j, \\ [h_i, e_j] = a_{ij}e_j, & [h_i, f_j] = -a_{ij}f_j, \\ (ad e_i)^{1-a_{ij}}e_j = (ad f_i)^{1-a_{ij}}f_j = 0, \text{ if } i \neq j. \end{cases} \quad (1.1)$$

The important property of the algebra  $\mathfrak{g}(A)$  is that it is simple or almost simple: it is simple after factorization by some known ideal.

We mention some general features of the theory of Kac–Moody algebras  $\mathfrak{g}(A)$ .

1. The symmetrization  $B$  defines a free  $\mathbb{Z}$ -module  $Q = \sum_{i \in I} \mathbb{Z}\alpha_i$  with generators  $\alpha_i, i \in I$ , equipped with symmetric bilinear form  $((\alpha_i, \alpha_j)) = B$  defined by the symmetrization  $B$ . The  $Q$  is called *root lattice*. The algebra  $\mathfrak{g}(A)$  is *graded by the root lattice*  $Q$  (by definition, generators  $h_i, e_i, f_i$  have weights 0,  $\alpha_i, (-\alpha_i)$  respectively):

$$\mathfrak{g}(A) = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha = \mathfrak{g}_0 \bigoplus \left( \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha \right) \bigoplus \left( \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_{-\alpha} \right) \quad (1.2)$$

where  $\mathfrak{g}_\alpha$  are finite dimensional linear spaces,  $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ ,  $\mathfrak{g}_0 \equiv Q \otimes \mathbb{C}$  is commutative, and is called *Cartan subalgebra*. An element  $0 \neq \alpha \in Q$  is called *root* if  $\mathfrak{g}_\alpha \neq 0$ . The  $\mathfrak{g}_\alpha$  is called *root space* corresponding to  $\alpha$ . The dimension  $\text{mult}(\alpha) = \dim \mathfrak{g}_\alpha$  is called *multiplicity* of the root  $\alpha$ . In (1.2),  $\Delta \subset Q$  is the set of all roots. It is divided in the set of positive  $\Delta_+ \subset \sum_{i \in I} \mathbb{Z}_+\alpha_i$  and negative  $-\Delta_+$  roots. A root  $\alpha \in \Delta$  is called *real* if  $(\alpha, \alpha) > 0$ . Otherwise, (if  $(\alpha, \alpha) \leq 0$ ), it is called *imaginary*. Every real root  $\alpha$  defines a *reflection*  $s_\alpha : x \mapsto x - (2(x, \alpha)/(\alpha, \alpha))\alpha$ ,  $x \in Q$ . All reflections  $s_\alpha$  in real roots generate *Weyl group*  $W \subset O(S)$ . The set of roots  $\Delta$  and multiplicities of roots are  $W$ -invariant.

2. One has the *Weyl–Kac denominator identity* which permits to calculate multiplicities of roots:

$$e(-\rho) \prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\text{mult}(\alpha)} = \sum_{w \in W} \det(w) e(-w(\rho)). \quad (1.3)$$

Here  $e(\cdot) \in \mathbb{Z}[Q]$  are formal exponents,  $\rho$  is called *Weyl vector* and is defined by the condition  $(\rho, \alpha_i) = -(\alpha_i, \alpha_i)/2$  for any  $i \in I$ .

The formula (1.3) is combinatorial, and formulae for multiplicities  $\text{mult}(\alpha)$  are unknown in general. One approach to solve this problem is to replace the formal function (1.3) by non-formal one (e. g. replacing formal exponents by non-formal ones) to get a function with “good” properties. These good properties may help to find the formulae for multiplicities.

**1.2. Finite and affine cases.** There are two cases when we have very clear picture (or Theory) of Kac-Moody algebras.

*Finite case:* The generalized Cartan matrix  $A$  is positive definite,  $A > 0$ . Then  $\mathfrak{g}(A)$  is finite-dimensional, and we get the classical theory of finite-dimensional semisimple Lie algebras.

*Affine case:* The generalized Cartan matrix  $A$  is semi-positive definite,  $A \geq 0$ . Then  $\mathfrak{g}(A)$  is called *affine*.

For both these cases we have three very nice properties:

(I). There exists the classification of all possible generalized Cartan matrices  $A$ : They are classified by Dynkin (for finite case) and by extended Dynkin (for affine case) diagrams.

(II). In the denominator identity (1.2), formal exponents may be replaced by non-formal ones to give a function with nice properties: For finite case this gives a polynomial. For affine case this gives a Jacobi automorphic form. Using these properties (or directly), one can find all multiplicities.

(III) Both these cases have extraordinary importance in Mathematics and Physics.

We want to construct similar Theory (to the theories of finite and affine Lie algebras above) for *Lorentzian (or hyperbolic) case* when the generalized Cartan matrix  $A$  is *hyperbolic*: it has exactly one negative square, all its other squares are either positive or zero. There are plenty of hyperbolic generalized Cartan matrices, it is impossible to find all of them and classify. On the other hand, probably not all of them give interesting Kac-Moody algebras, and one has to find natural conditions on these matrices.

**1.3. Lorentzian case. Example of R. Borcherds.** We have the following *key example due to R. Borcherds* [B1]—[B5].

For Borcherds example, the root lattice  $Q = S$  where  $S$  is an even hyperbolic unimodular lattice  $S$  of signature  $(25, 1)$ . Here “even” means that  $(x, x)$  is even for any  $x \in S$ . “Unimodular” means that the dual lattice  $S^*$  coincides with  $S$ , equivalently, for a bases  $e_1, \dots, e_{26}$  of  $S$  the determinant of the Gram matrix  $((e_i, e_j))$  is equal to  $\pm 1$ . A lattice  $S$  with these properties is unique up to isomorphism. For Borcherds example, the Weyl group  $W$  is generated by reflections  $s_\alpha : x \mapsto x - (x, \alpha)\alpha$ ,  $x \in S$ , in all elements  $\alpha \in S$  with  $\alpha^2 = 2$ . The group  $W$  is discrete in the hyperbolic space  $\mathcal{L}(S) = V^+(S)/\mathbb{R}_{++}$ . Here  $V^+(S)$  is a half of the light cone  $V(S) = \{x \in S \otimes \mathbb{R} \mid x^2 < 0\}$  of  $S$ . The  $\mathcal{L}(S)$  is the set of rays in  $V^+(S)$ .

A fundamental chamber  $\mathcal{M} \subset \mathcal{L}(S)$  for  $W$  is defined by the set  $P$  of elements  $\alpha \in S$  with  $\alpha^2 = 2$  which are *orthogonal to  $\mathcal{M}$* . It has the following description due to J. Conway [Co]. There exists an orthogonal decomposition  $S = [\rho, e] \oplus L$  where the Gram matrix of elements  $\rho, e$  is equal to  $U = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$  (in particular,  $(\rho, \rho) = 0$ ), and  $L$  is the Leech lattice, i. e. positive definite even unimodular lattice of the rank 24 without elements with square 2. The set  $P$  of roots which are orthogonal to the fundamental chamber  $\mathcal{M}$  (or the set of *simple roots*) of  $W$  is equal to

$$P = \{\alpha \in S \mid (\alpha, \alpha) = 2 \quad \& \quad (\rho, \alpha) = -1\}. \quad (1.4)$$

It means that the fundamental chamber  $\mathcal{M} \subset \mathcal{L}(S)$  is equal to

$$\mathcal{M} = \{\mathbb{R}_{++} \alpha \in \mathcal{L}(S) \mid (\alpha, \rho) < 0\} \quad (1.5)$$

and  $P$  is a minimal set with this property. We mention that the fundamental polyhedron  $\mathcal{M}$  has "*almost finite*" volume. It means that  $\mathcal{M}$  is finite in any angle with the center at infinity  $\mathbb{R}_{++}\rho$  of the hyperbolic space  $\mathcal{L}(S)$ .

The matrix

$$A = ((\alpha, \alpha')), \quad \alpha, \alpha' \in P \quad (1.6)$$

is a generalized Cartan matrix and  $\rho$  is the Weyl vector:

$$(\rho, \alpha) = -(\alpha, \alpha)/2, \quad \forall \alpha \in P. \quad (1.7)$$

We have the classical  $SL_2(\mathbb{Z})$ -modular cusp form  $\Delta$  of the weight 12 on the upper-half plane in  $q > 0$ :

$$\Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{m \geq 0} \tau(m) q^m, \quad (1.8)$$

where  $q = \exp(2\pi i\tau)$ . We have

$$\Delta^{-1} = \sum_{n \geq 0} p_{24}(n) q^{n-1} \quad (1.9)$$

where  $p_{24}(n)$  are positive integers. Borcherds [B2] proved the *identity*

$$\begin{aligned} \Phi(z) = \exp(-2\pi i(\rho, z)) \prod_{\alpha \in \Delta_+} (1 - \exp(-2\pi i(\alpha, z)))^{p_{24}(1 - (\alpha, \alpha)/2)} = \\ \sum_{w \in W} \det(w) \sum_{m > 0} \tau(m) \exp(-2\pi i(w(m\rho), z)). \end{aligned} \quad (1.10)$$

Here  $\Delta_+ = \{\alpha \in S \mid \alpha^2 = 2 \ \& \ (\alpha, \rho) < 0\} \cup (S \cap \overline{V^+(S)} - \{0\})$ . The variable  $z$  runs through the *complexified cone*  $\Omega(V^+(S)) = S \otimes \mathbb{R} + iV^+(S)$  of the light cone  $V^+(S)$ . Moreover, Borcherds [B4], [B5] proved that the function  $\Phi(z)$  is an *automorphic form of weight 12* with respect to the group  $O^+(T)$  where  $T = U \oplus S$  is the extended lattice of the signature  $(26, 2)$ . The group  $O^+(T)$  naturally acts in the Hermitian symmetric domain of type IV

$$\Omega(T) = \{\mathbb{C}\omega \subset T \otimes \mathbb{C} \mid (\omega, \omega) = 0 \ \& \ (\omega, \overline{\omega}) < 0\}_0, \quad (1.11)$$

which has canonical identification with  $\Omega(V^+(S))$  as follows:  $z \in \Omega(V^+(S))$  defines the element  $\mathbb{C}\omega_z \in \Omega(T)_0$  where  $\omega_z = ((z, z)/2)e_1 + e_2 \oplus z \in T \otimes \mathbb{C}$  and  $e_1, e_2$  is the bases of the lattice  $U$  with the Gram matrix  $U$  above. Here "automorphic of the weight 12" means that the function  $\widetilde{\Phi}(\lambda\omega_z) = \lambda^{-12}\widetilde{\Phi}(z)$ ,  $\lambda \in \mathbb{C}^*$ , is homogeneous of the degree  $-12$  (it is obvious) in the cone  $\widetilde{\Omega(T)}_0$  over  $\Omega(T)_0$ , and  $\widetilde{\Phi}(g\omega) = \widetilde{\Phi}(\omega)$  for any  $g \in O^+(T)$  where  $O^+(T)$  is the subgroup of index 2 of the group  $O(T)$  which keeps the connected component (1.11) (marked by 0).

The identity (1.10) looks very familiar to the form (1.3) of the denominator identity for Kac–Moody algebras, but it has some difference.

To interpret (1.10) as a denominator identity of a Lie algebra, Borcherds introduced [B1] *generalized Kac–Moody algebras*  $\mathfrak{g}(A')$  which correspond to more general matrices  $A'$  than generalized Cartan matrices. Here I will call them *generalized*

*generalized Cartan matrices.* Difference is that a generalized generalized Cartan matrix  $A'$  may also have non-positive real elements  $a_{ij} \leq 0$  on the diagonal and out of the diagonal, but all  $a_{ij} \in \mathbb{Z}$  if  $a_{ii} = 2$ . A definition of the generalized Kac-Moody algebra  $\mathfrak{g}(A')$  corresponding to a generalized generalized Cartan matrix  $A'$  is similar to (1.1). One should replace the last line of (1.1) by

$$(ad e_i)^{1-a_{ij}} e_j = (ad f_i)^{1-a_{ij}} f_j = 0 \text{ if } i \neq j \text{ and } a_{ii} = 2, \quad (1.12)$$

and add the relation

$$[e_i, e_j] = [f_i, f_j] = 0 \text{ if } a_{ij} = 0. \quad (1.13)$$

Borcherds showed that generalized Kac-Moody algebras have similar properties to ordinary Kac-Moody algebras. They also have a denominator identity which has more general form than (1.3) and includes (1.10) as a particular case.

The identity (1.10) is the denominator identity for the generalized Kac-Moody algebra  $\mathfrak{g}(A')$  where  $A'$  is the generalized generalized Cartan matrix equals to the Gram matrix  $A' = ((\alpha, \alpha'))$ ,  $\alpha, \alpha' \in P'$  where

$$P' = P \cup 24\rho \cup 24(2\rho) \cup \dots \cup 24(n\rho) \cup \dots \quad (1.14)$$

is the sequence of elements of the lattice  $S$ . Here  $24(n\rho)$  means that we take the element  $n\rho$  twenty four times to get the Gram matrix  $A'$ . See details in [B1]—[B3].

In (1.14), the set  $P'$  defining  $A'$  is called the *set of simple roots*. It is divided in the set  $P'^{re} = P$  of *simple real roots* (they are orthogonal to the fundamental chamber  $\mathcal{M}$  of the Weyl group  $W$  and have positive square) and is the same as for the ordinary Kac-Moody algebra  $\mathfrak{g}(A)$  defined by the generalized Cartan matrix  $A$  in (1.6). The additional sequence

$$P'^{im} = 24\rho \cup 24(2\rho) \cup \dots \cup 24(n\rho) \cup \dots \quad (1.15)$$

of  $P'$  (elements of  $P'^{im}$  have non-positive square) is defined by the Fourier coefficients in the sum part of the identity (1.10). For example,  $24$  is defined by the  $24$  in (1.8). Together  $P'^{re}$  and  $P'^{im}$  define the generalized generalized Cartan matrix  $A'$  and the generalized Kac-Moody algebra  $\mathfrak{g}(A')$ .

Borcherds example is very fundamental and beautiful. It has several applications in Mathematics (e.g. for the Monster) and Physics (e.g. for String Theory).

**1.4. A Theory of Lorentzian Kac-Moody algebras.** Analizing Borcherds example, one can suggest a general class of Lorentzian Kac-Moody algebras (or automorphic Kac-Moody algebras)  $\mathfrak{g}$ , see [N7], [N8], [GN1]—[GN7].

One takes *data* (1)—(5) below:

(1) A *hyperbolic lattice*  $S$  (i. e. an integral symmetric bilinear form of signature  $(n, 1)$ ).

(2) A *reflection group* (or *Weyl group*)  $W \subset O(S)$  generated by reflections in roots of  $S$ . We remind that  $\alpha \in S$  is called *root* if  $\alpha^2 > 0$  and  $\alpha^2 \mid 2(\alpha, S)$ . Any root defines a reflection  $s_\alpha : x \mapsto x - (2(x, \alpha)/\alpha^2)\alpha$ ,  $x \in S$ , which gives an automorphism of the lattice  $S$ .

(3) A *set*  $P$  of *orthogonal roots* to a *fundamental chamber*  $\mathcal{M} \subset \mathcal{L}(S)$  of  $W$ . It means that the set  $P$  of roots of  $S$  should have the property (1.5) and should be minimal having this property. Moreover, the set  $P$  should have a *Weyl vector*

$\rho \in S \otimes \mathbb{Q}$  defined by the equality (1.7) (it is more right to call it as *lattice Weyl vector*).

The main invariant of the data (1)—(3) is the *generalized Cartan matrix*

$$A = \left( \frac{2(\alpha, \alpha')}{(\alpha, \alpha)} \right), \quad \alpha, \alpha' \in P. \quad (1.16)$$

It defines data (1)—(3) up to some very clear equivalence and defines the set of real roots of the algebra  $\mathfrak{g}$  we want to construct.

(4) An *automorphic (holomorphic) form*  $\Phi(z)$  on a IV type Hermitian symmetric domain,  $z \in \Omega(V^+(S)) = \Omega(T)$ , with respect to a subgroup  $G \subset O^+(T)$  of finite index of an extended lattice  $T = U(k) \oplus S$  where  $U(k) = \begin{pmatrix} 0 & -k \\ -k & 0 \end{pmatrix}$ ,  $k \in \mathbb{N}$ . (See [GN5] for more general definition.) It should have Fourier expansion of the form of the denominator identity for a generalized Kac–Moody algebra with hyperbolic generalized Cartan matrix. This form is

$$\Phi(z) = \sum_{w \in W} \det(w) \left( \exp(-2\pi i(w(\rho), z)) - \sum_{a \in S \cap \mathbb{R}_{++} \mathcal{M}} m(a) \exp(-2\pi i(w(\rho + a), z)) \right) \quad (1.17)$$

where all coefficients  $m(a)$  should be integral. The automorphic form  $\Phi$  defines the set of imaginary roots of the algebra  $\mathfrak{g}$ .

Like for Borchers example, data (1)—(4) define a *generalized Kac–Moody algebra or superalgebra* (if some Fourier coefficients  $m(a)$  are negative)  $\mathfrak{g}$ . See the definition of  $\mathfrak{g}$  below. Using automorphic properties of  $\Phi(z)$ , it is good to *calculate the product part of the denominator identity*

$$\Phi(z) = \exp(-2\pi i(\rho, z)) \prod_{\alpha \in \Delta_+} \left( 1 - \exp(-2\pi i(\alpha, z)) \right)^{\text{mult}(\alpha)} \quad (1.18)$$

which gives multiplicities  $\text{mult}(\alpha)$  of roots  $\alpha$  of  $\mathfrak{g}$ . For superalgebras case, the  $\text{mult}(\alpha) = \dim \mathfrak{g}_{\alpha, \bar{0}} - \dim \mathfrak{g}_{\alpha, \bar{1}}$  is the difference of dimensions of even and odd parts of the root space  $\mathfrak{g}_\alpha$ .

It was understood that it is natural to suppose (at least, to have finiteness results) the additional condition:

(5) The automorphic form  $\Phi$  on the domain  $\Omega(V^+(S)) = \Omega(T)$  should be *reflective*. It means that the divisor (of zeros) of  $\Phi$  is union of quadratic divisors orthogonal to roots of the extended lattice  $T$ . Here for a root  $\alpha \in T$  (the definition of a root of  $T$  is the same as for the lattice  $S$ ) the *quadratic divisor orthogonal to  $\alpha$*  is equal to

$$D_\alpha = \{\mathbb{C}\omega \in \Omega(T) \mid (\omega, \alpha) = 0\}. \quad (1.19)$$

The property (5) is valid for Borchers example above and in all known cases. Moreover, it is true in the neighbourhood of the cusp where the product (1.18) converges. Thus, we want it to be true globally.

A generalized Kac–Moody superalgebra  $\mathfrak{g}$  corresponding to data (1)—(4) is given by the sequence  $P' \subset S$  of *simple roots*. This sequence is divided in a set  $P'^{re}$  of *simple real roots* and a sequence  $P'^{im}$  of *simple imaginary roots*. The sequence  $P'^{im}$  is divided in the sequence  $P'^{im}_{\bar{0}}$  of *even simple imaginary roots* and a sequence  $P'^{im}_{\bar{1}}$  of *odd simple imaginary roots*.

of *odd simple imaginary roots*. For any primitive  $a \in S \cap \mathbb{R}_{++}\mathcal{M}$  with  $(a, a) = 0$ , one should find  $\tau(na) \in \mathbb{Z}$ ,  $n \in \mathbb{N}$ , from the identity with the formal variable  $T$ :

$$1 - \sum_{k \in \mathbb{N}} m(ka) T^k = \prod_{n \in \mathbb{N}} (1 - T^n)^{\tau(na)}.$$

We set  $P'^{re} = P$  where  $P$  is defined in (3). The set  $P'^{re}$  is considered to be even:  $P'^{re} = P'^{re}_0$  and  $P'^{re}_1 = \emptyset$ . We set

$$P'^{im}_0 = \{m(a)a \mid a \in S \cap \mathbb{R}_{++}\mathcal{M}, (a, a) < 0 \text{ and } m(a) > 0\} \cup \\ \cup \{\tau(a)a \mid a \in S \cap \mathbb{R}_{++}\mathcal{M}, (a, a) = 0 \text{ and } \tau(a) > 0\};$$

$$P'^{im}_1 = \{-m(a)a \mid a \in S \cap \mathbb{R}_{++}\mathcal{M}, (a, a) < 0 \text{ and } m(a) < 0\} \cup \\ \cup \{\tau(a)a \mid a \in S \cap \mathbb{R}_{++}\mathcal{M}, (a, a) = 0 \text{ and } \tau(a) < 0\}.$$

The generalized Kac-Moody superalgebra  $\mathfrak{g}$  is a Lie superalgebra generated by  $h_r, e_r, f_r$  where  $r \in P'$ . All generators  $h_r$  are even, generators  $e_r, f_r$  are even (respectively odd) if  $r$  is even (respectively odd). They have the defining relations 1) — 5) below:

- 1) The map  $r \mapsto h_r$  for  $r \in P'$  gives an embedding of  $S \otimes \mathbb{C}$  to  $\mathfrak{g}$  as an abelian subalgebra (it is even).
- 2)  $[h_r, e_{r'}] = (r, r')e_{r'}$  and  $[h_r, f_{r'}] = -(r, r')f_{r'}$ .
- 3)  $[e_r, f_{r'}] = h_r$  if  $r = r'$ , and is 0 if  $r \neq r'$ .
- 4)  $(ad e_r)^{1-2(r, r')/(r, r)}e_{r'} = (ad f_r)^{1-2(r, r')/(r, r)}f_{r'} = 0$  if  $r \neq r'$  and  $(r, r) > 0$  (equivalently,  $r \in P'^{re}$ ).
- 5) If  $(r, r') = 0$ , then  $[e_r, e_{r'}] = [f_r, f_{r'}] = 0$ .

See [B1], [GN1], [GN2], [GN5], [R] for details. We remark that this definition is equivalent (for ordinary generalized Kac-Moody algebras) to the definition above using a generalized Cartan matrix defined by the Gram matrix of the sequence  $P'$ .

*The generalized Kac-Moody superalgebras  $\mathfrak{g}$  above given by the data (1)–(5) constitute the Theory of Lorentzian Kac-Moody algebras (or automorphic Lorentzian Kac-Moody algebras) which we consider.*

By (4), they have similar property to the property (II) for finite and affine algebras: their denominator identity gives an automorphic form. For Lorentzian case, it is an automorphic form on IV type symmetric domain.

What about a similar property to the property (I) for finite and affine algebras? How many data (1)–(5) one may have?

Further, we suppose that  $\text{rk } S \geq 3$ . This condition is equivalent to considering hyperbolic analogues of non-trivial finite-dimensional semi-simple Lie algebras. When  $\text{rk } S = 1, 2$ , classification problem of data (1)–(5) is different and, it seems, more simple.

We have (see [N3], [N4], [N7], [N8], [GN5]):

**Theorem 1.** *If  $\text{rk } S \geq 3$ , the set of possible data (1)–(3) in data (1)–(4) is finite when  $(\rho, \rho) < 0$  and is in essential finite when  $(\rho, \rho) = 0$ . The inequality  $(\rho, \rho) > 0$  is impossible.*

In the theorem, “in essential finite” means that the set might be infinite, but we have very clear understanding of the set. For example, the set of possible Dynkin diagrams of the type  $A_n$  is infinite, but we understand the set very clearly.

Key point of the proof of Theorem 1 is that (1)—(4) imply that  $(\rho, \rho) \leq 0$  and the fundamental chamber  $\mathcal{M}$  has finite (if  $(\rho, \rho) < 0$ ) or almost finite (if  $(\rho, \rho) = 0$ ) volume, see [N8], [GN5]. (Here “almost finite” means the same as for the Borchers example above.) Then the number of possible root lattices  $S$  is finite. It follows from old results of author [N3], [N4] (and also [N8]) and É.B. Vinberg [V]. If there additionally exists a Weyl vector  $\rho$  for  $P$ , one also has finiteness (when  $(\rho, \rho) < 0$ ) or in essential finiteness (when  $(\rho, \rho) = 0$ ) of the sets of Weyl groups  $W$ , fundamental chambers  $\mathcal{M}$  (up to action of  $W$ ) and the sets of orthogonal roots  $P$  to  $\mathcal{M}$ , see [N8]. This gives finiteness or in essential finiteness of possible generalized Cartan matrices  $A$  in (1.16) corresponding to systems of simple real roots.

It follows that in principle we can classify all possible data (1)—(3) in the data (1)—(4). It makes the Theory of Lorentzian Kac–Moody algebras similar to the Theories of finite and affine algebras.

It would be nice to have also finiteness results for the data (4), (5). Recently, here, there were obtained some partial finiteness results [N9], [GN5] which show that automorphic forms  $\Phi(z)$  in (4) and (5) are extremely rare. It makes very possible the following statement:

**Conjecture 2.** *If  $\text{rk } S \geq 3$ , the set of possible data (4), (5) is in essential finite.*

The reason why we expect the statement of Conjecture 2, is based on the *Koecher principle* (e. g. see [Ba]): Any holomorphic automorphic form on a Hermitian symmetric domain  $\Omega$  should have zeros in  $\Omega$  if  $\dim \Omega - \dim \Omega_\infty \geq 2$ .

Applying this principle to restrictions  $\Phi|_{\Omega(T_1)}$  of a reflective automorphic form  $\Phi$  on all subdomains  $\Omega(T_1) \subset \Omega(T)$  where  $T_1 \subset T$  is a sublattice of  $T$  of signature  $(k, 2)$ , one obtains very strong restrictions on the lattice  $T$  to have a reflective automorphic form  $\Phi$ . This was shown in [N9].

We think that Conjecture 2 is very interesting. From our point of view, the theory of reflective automorphic forms on IV type domains  $\Omega(T)$  where  $T$  is a lattice of signature  $(n, 2)$  is similar (Mirror Symmetric) to the theory of reflection groups  $W$  of hyperbolic lattices  $S$  with fundamental chamber of finite or almost finite volume, see [GN3], [GN5]—[GN7], [N9].

It is interesting to classify (or describe) the conjecturally “finite set” of data (1)—(5). Even a finite set may have very interesting structure. As a result, we will have a Theory of Lorentzian Kac–Moody algebras which one can consider as a hyperbolic analogy of the Theories of finite and affine Kac–Moody algebras.

At the end, we describe a small piece of this classification which was obtained in [GN4]—[GN6].

There are exactly 12 generalized Cartan matrices  $A$  of data (1)—(3) in (1)—(4) which are symmetric, have the rank 3 and have the Weyl vector  $\rho$  with  $(\rho, \rho) < 0$ . They are given below:

**The list of all symmetric hyperbolic generalized Cartan matrices of the rank 3 with  $\text{vol}(\mathcal{M}) < \infty$  and a lattice Weyl vector  $\rho$**

$$A_{1,0} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -2 \\ -1 & -2 & 2 \end{pmatrix}, \quad A_{1,I} = \begin{pmatrix} 2 & -2 & -1 \\ -2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \quad A_{1,II} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix},$$

$$A_{1,III} = \begin{pmatrix} 2 & -2 & -6 & -6 & -2 \\ -2 & 2 & 0 & -6 & -7 \\ -6 & 0 & 2 & -2 & -6 \\ -6 & -6 & -2 & 2 & 0 \end{pmatrix}; \quad A_{2,0} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix}, \quad A_{2,I} = \begin{pmatrix} 2 & -2 & -4 & 0 \\ -2 & 2 & 0 & -4 \\ -4 & 0 & 2 & -2 \\ 0 & 4 & 2 & 2 \end{pmatrix},$$



$$A_{2,II} = \begin{pmatrix} 2 & -2 & -6 & -2 \\ -2 & 2 & -2 & -6 \\ -6 & -2 & 2 & -2 \\ -2 & -6 & -2 & 2 \end{pmatrix}, \quad A_{2,III} = \begin{pmatrix} 2 & -2 & -8 & -16 & -18 & -14 & -8 & 0 \\ -2 & 2 & 0 & -8 & -14 & -18 & -16 & -8 \\ -8 & 0 & 2 & -2 & -8 & -16 & -18 & -14 \\ -16 & -8 & -2 & 2 & 0 & -8 & -14 & -18 \\ -18 & -14 & -8 & 0 & 2 & -2 & -8 & -16 \\ -14 & -18 & -16 & -8 & -2 & 2 & 0 & -8 \\ -8 & -16 & -18 & -14 & -8 & 0 & 2 & -2 \\ 0 & -8 & -14 & -18 & -16 & -8 & -2 & 2 \end{pmatrix};$$

$$A_{3,0} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -1 \\ -2 & -1 & 2 \end{pmatrix}, \quad A_{3,I} = \begin{pmatrix} 2 & -2 & -5 & -1 \\ -2 & 2 & -1 & -5 \\ -5 & -1 & 2 & -2 \\ -1 & -5 & -2 & 2 \end{pmatrix},$$

$$A_{3,II} = \begin{pmatrix} 2 & -2 & -10 & -14 & -10 & -2 \\ -2 & 2 & -2 & -10 & -14 & -10 \\ -10 & -2 & 2 & -2 & -10 & -14 \\ -14 & -10 & -2 & 2 & -2 & -10 \\ -10 & -14 & -10 & -2 & 2 & -2 \\ -2 & -10 & -14 & -10 & -2 & 2 \end{pmatrix},$$

$$A_{3,III} = \begin{pmatrix} 2 & -2 & -11 & -25 & -37 & -47 & -50 & -46 & -37 & -23 & -11 & -1 \\ -2 & 2 & -1 & -11 & -23 & -37 & -46 & -50 & -47 & -37 & -25 & -11 \\ -11 & -1 & 2 & -2 & -11 & -25 & -37 & -47 & -50 & -46 & -37 & -23 \\ -25 & -11 & -2 & 2 & -1 & -11 & -23 & -37 & -46 & -50 & -47 & -37 \\ -37 & -23 & -11 & -1 & 2 & -2 & -11 & -25 & -37 & -47 & -50 & -46 \\ -47 & -37 & -25 & -11 & -2 & 2 & -1 & -11 & -23 & -37 & -46 & -50 \\ -50 & -46 & -37 & -23 & -11 & -1 & 2 & -2 & -11 & -25 & -37 & -47 \\ -46 & -50 & -47 & -37 & -25 & -11 & -2 & 2 & -1 & -11 & -23 & -37 \\ -37 & -47 & -50 & -46 & -37 & -23 & -11 & -1 & 2 & -2 & -11 & -25 \\ -23 & -37 & -46 & -50 & -47 & -37 & -25 & -11 & -2 & 2 & -1 & -11 \\ -11 & -25 & -37 & -47 & -50 & -46 & -37 & -23 & -11 & -1 & 2 & -2 \\ -1 & -11 & -23 & -37 & -46 & -50 & -47 & -37 & -25 & -11 & -2 & 2 \end{pmatrix}.$$

For all these cases the fundamental chamber  $\mathcal{M}$  is a closed polygon on the hyperbolic plane with angles respectively:

$A_{1,0} : \pi/2, 0, \pi/3$ ;  $A_{1,I} : 0, \pi/3, \pi/3$ ;  $A_{1,II} : 0, 0, 0$ ;  $A_{1,III} : 0, \pi/2, 0, \pi/2, 0$ ;

$A_{2,0} : 0, \pi/2, 0$ ;  $A_{2,I} : 0, \pi/2, 0, \pi/2$ ;  $A_{2,II} : 0, 0, 0, 0$ ;

$A_{2,III} : 0, \pi/2, 0, \pi/2, 0, \pi/2, 0, \pi/2$ ;

$A_{3,0} : 0, \pi/3, 0$ ;  $A_{3,I} : 0, \pi/3, 0, \pi/3$ ;  $A_{3,II} : 0, 0, 0, 0, 0, 0$ ;

$A_{3,III} : 0, \pi/3, 0, \pi/3, 0, \pi/3, 0, \pi/3, 0, \pi/3, 0, \pi/3$ .

All these polygons are touching a circle with the center  $\mathbb{R}_{++}\rho$  where  $\rho$  is the Weyl vector. It shows the geometrical sense of the Weyl vector.

For 9 generalized Cartan matrices  $A_{i,j}$  where  $i = 1, 2, 3$  and  $j = 0, I, II$  we found automorphic forms  $\Phi$  for data (4), (5), found their product formulae (1.18), and thus constructed the corresponding (automorphic) Lorentzian Kac-Moody algebras. See [GN1], [GN2], [GN4]—[GN6]. It is interesting that some of these automorphic forms were well-known. For example, the automorphic form  $\Phi$  for  $A_{1,II}$  is classical. It has the weight 5 and is the product of all even theta-constants of genus 2 (there are 10 of them). It is automorphic with respect to  $Sp_4(\mathbb{Z})$  with some quadratic character and gives the discriminant of curves of genus 2. The automorphic form  $\Phi$  for  $A_{1,0}$  has the weight 35 and is automorphic with respect to  $Sp_4(\mathbb{Z})$ . This automorphic form was found by Igusa 30 years ago and is  $Sp_4(\mathbb{Z})$ -automorphic form of the smallest odd weight. For both these automorphic forms, we found their product expansions (1.18) which were not known. Here we use isomorphism of IV type symmetric domain of dimension 3 with Siegel upper-half plane of genus 2.

All other automorphic forms  $\Phi$  for generalized Cartan matrices  $A_{1,0}$ — $A_{3,II}$  were not known. Let us give one of them.

We give the automorphic form  $\Phi$  for  $A_{3,II}$ . For this case  $T = 2U(12) \oplus \langle 2 \rangle = U(12) \oplus S$  where  $S = U(12) \oplus \langle 2 \rangle$  gives the datum (1). We use bases  $f_2, \hat{f}_3, f_{-2}$  of the lattice  $S$  with the Gram matrix

$$\begin{pmatrix} 0 & 0 & -12 \\ 0 & 2 & 0 \\ -12 & 0 & 0 \end{pmatrix}$$

and corresponding coordinates  $z_1, z_2, z_3$  of  $S \otimes \mathbb{C}$ . The Weyl group  $W$  in (2) is generated by reflections in all elements with square 2 of  $S$ . The set  $P$  in (3) equals

$$P = \{\alpha_1 = (0, 1, 0), \alpha_2 = (0, -1, 1), \alpha_3 = (1, -5, 2), \alpha_4 = (2, -7, 2), \\ \alpha_5 = (2, -5, 1), \alpha_6 = (1, -1, 0)\}.$$

It has the Gram matrix  $A_{3,II}$ . The Weyl vector  $\rho = (\frac{1}{6}, -\frac{1}{2}, \frac{1}{6})$ . The automorphic form  $\Phi$  is an automorphic cusp form  $\Delta_1$  with respect to the group  $G = O^+(T)$  with some character of order 6. It has the smallest possible weight 1 and has the Fourier expansion and the product expansion

$$\begin{aligned} \Delta_1(z_1, z_2, z_3) &= \sum_{M \geq 1} \sum_{\substack{m > 0, l \in \mathbb{Z} \\ n, m \equiv 1 \pmod{6} \\ 4nm - 3l^2 = M^2}} \left(\frac{-4}{l}\right) \left(\frac{12}{M}\right) \sum_{a|(n,l,m)} \left(\frac{6}{a}\right) q^{n/6} r^{l/2} s^{m/6} \\ &= q^{1/6} r^{1/2} s^{1/6} \prod_{\substack{n, l, m \in \mathbb{Z} \\ (n, l, m) > 0}} (1 - q^n r^l s^m)^{f_3(nm, l)} \end{aligned} \quad (1.20)$$

where  $q = \exp(24\pi i z_1)$ ,  $r = \exp(4\pi i z_2)$ ,  $s = \exp(24\pi i z_3)$  and

$$\begin{aligned} \left(\frac{-4}{l}\right) &= \begin{cases} \pm 1 & \text{if } l \equiv \pm 1 \pmod{4} \\ 0 & \text{if } l \equiv 0 \pmod{2} \end{cases}, \quad \left(\frac{12}{M}\right) = \begin{cases} 1 & \text{if } M \equiv \pm 1 \pmod{12} \\ -1 & \text{if } M \equiv \pm 5 \pmod{12} \\ 0 & \text{if } (M, 12) \neq 1 \end{cases}, \\ \left(\frac{6}{a}\right) &= \begin{cases} \pm 1 & \text{if } a \equiv \pm 1 \pmod{6} \\ 0 & \text{if } (a, 6) \neq 1 \end{cases}. \end{aligned}$$

The multiplicities  $f_3(nm, l)$  of the infinite product are defined by a weak Jacobi form  $\phi_{0,3}(\tau, z) = \sum_{n \geq 0, l \in \mathbb{Z}} f_3(n, l) q^n r^l$  of weight 0 and index 3 with integral Fourier coefficients:

$$\phi_{0,3}(\tau, z) = r^{-1} \left( \prod_{n \geq 1} (1 + q^{n-1} r) (1 + q^n r^{-1}) (1 - q^{2n-1} r^2) (1 - q^{2n-1} r^{-2}) \right)^2 \quad (1.21)$$

where  $q = \exp(2\pi i \tau)$ ,  $\text{im } \tau > 0$ , and  $r = \exp(2\pi i z)$ . The divisor of  $\Delta_1$  is the sum with multiplicities one of all quadratic divisors orthogonal to elements of  $T$  with square 2. The  $S, W, P$  and  $\Delta_1$  define the generalized Lorentzian Kac–Moody superalgebra  $\mathfrak{g}$  with the denominator identity (1.20).

To construct the automorphic form  $\Delta_1$ , we use the *arithmetic lifting of Jacobi forms on IV type symmetric domains* constructed in [G1]–[G4] (it gives the sum part of (1.20)). To find the product part of (1.20), we use the *Borcherds lifting* [B5] which is the exponential analog of the arithmetic lifting. See details in [GN6].

**1.5. Physical applications.** Lorentzian Kac-Moody algebras we have considered above and corresponding automorphic forms  $\Phi$  found some very interesting applications in Physics: String Theory, Mirror Symmetry and others. We refer a reader to a nice review [M] and references there. E. g. see [Ca], [CCL], [DVV], [HM1]—[HM2], [Ka1]—[Ka2]. Roughly speaking the Lorentzian Kac-Moody algebras are related with symmetries of Fundamental Physical Theories.

### 1.6. An interesting problem.

In data (1)—(5) above, existence of the Weyl vector  $\rho \in S \otimes \mathbb{Q}$  (or the lattice Weyl vector) is very important. It is equivalent considering automorphic forms  $\Phi$  on IV type Hermitian symmetric domains. It would be interesting to extend the Theory of Lorentzian Kac-Moody algebras above to cases when the lattice Weyl vector  $\rho$  does not exist. It seems, for this more general case, one should consider automorphic forms in some more general sense. On the other hand, this more general theory will lose some finiteness properties. It would be a pity.

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